Module B6

Matrices

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Introduction

Suppose we want to produce a model to describe and forecast economic behaviour. There are many variables involved in the modelling process. Economists look at such things as aggregate demand, consumption, investment, supply and demand and may obtain a set of equations such as the ones below:

$$1Y - 1C - 1I + 0R = \overline{G}$$

$$-bY + 1C + 0I + 0R = a$$

$$-dY + 0C + 1I - eR = 0$$

$$1Y + 0C + 0I + \frac{g}{f} \times R = \frac{1}{f} \times \overline{M}$$

With your knowledge of solving algebraic equations, this would be a fairly difficult process. With the help of matrices, the solution of 4 equations with 4 unknowns (or even more) becomes much simpler.

On completion of this module you should be able to:

- demonstrate an understanding of the form of a matrix by converting numerical data (narrative or table) into matrix form
- perform matrix operations including addition, subtraction, multiplication by a constant, and multiplication of two matrices
- describe the characteristics of a matrix including elements, order, equal matrices, and the identity matrix
- use matrices to solve real world problems
- demonstrate an understanding of the meaning of an inverse matrix
- use matrices and matrix methods to solve a set of simultaneous equations.

6.1 What are matrices?

6.1.1 Tables to matrices

In our everyday lives we often represent data in table form to make it meaningful and easy to read, e.g. cricket scores, the milkman's weekly order, an inventory for a tyre retailer.

Often this information is tabulated in the form of a *matrix* (plural is *matrices*). The word matrix originally came from the Latin meaning *womb* – that is, from which something originates. In mathematics its meaning is a table of rows and columns, but it has extremely wide usage in science, engineering, mathematics and business. For example, it is used in:

- business, especially in planning and production
- studying vibrations in car engines
- solving linear equations
- developing quantum theory
- in many circumstances where there are rotations, reflections or other distortions of geometrical figures such as in the theory of building construction, in electricity and magnetism and in aerodynamics.

Tables are an excellent way of displaying data but come in a range of forms so it is difficult to perform operations on them. With numbers we have a standard way of depicting them, for example, the number '4' (not usually IV, 10_4 or ||||) so that we can manipulate these numbers more easily (e.g. add, subtract, multiply, square). This is the same with tables. If we standardise them into matrices we now have a much more powerful tool to manipulate these tables.

Look carefully at the examples below. Can you see the main steps in converting tables to matrices?

Example

Each of the four mathematics courses in the Tertiary Preparation Program has an introductory book, a study book 1 and a study book 2. The following table shows the number of pages in each book. Express this as a matrix.

	Level A	Level B	Level C	Level D
Introductory book	72	75	104	62
Study book 1	380	359	320	350
Study book 2	320	308	226	280

$$\text{The matrix} = \begin{pmatrix} 72 & 75 & 104 & 62 \\ 380 & 359 & 320 & 350 \\ 320 & 308 & 226 & 280 \end{pmatrix}$$

Example

The Tooth Rot Sweet Company has a home delivery service that includes five types of packaged sweets. Create a matrix for a weekly order form from the table below.

	Mon	Tues	Wed	Thurs	Fri	Sat
500 g Family Assorted	150	120	200	100	300	350
1 kg Budget	100	100	150	50	120	200
250 g Mints	80	50	50	100	50	100
200 g Snakes	50	40	80	100	50	80
100 g Choc Balls	100	80	200	200	150	300

$$\text{The matrix} = \begin{pmatrix} 150 & 120 & 200 & 100 & 300 & 350 \\ 100 & 100 & 150 & 50 & 120 & 200 \\ 80 & 50 & 50 & 100 & 50 & 100 \\ 50 & 40 & 80 & 100 & 50 & 80 \\ 100 & 80 & 200 & 200 & 150 & 300 \end{pmatrix}$$

As you can see from the above examples a matrix is just like a table except the row and column headings are missing and there is a set of brackets around the numbers. Sometimes you may see row and column headings and labels as well. For example:

$$\begin{array}{ll} & \qquad\qquad\text{Days} \\ \text{Types of sweets} & \begin{array}{r} \\ \text{Family Ass} \\ \text{Budget} \\ \text{Mints} \\ \text{Snakes} \\ \text{Choc Balls} \end{array} \begin{array}{c} \begin{array}{cccccc} M & T & W & T & F & S \end{array} \\ \begin{pmatrix} 150 & 120 & 200 & 100 & 300 & 350 \\ 100 & 100 & 150 & 50 & 120 & 200 \\ 80 & 50 & 50 & 100 & 50 & 100 \\ 50 & 40 & 80 & 100 & 50 & 80 \\ 100 & 80 & 200 & 200 & 150 & 300 \end{pmatrix} \end{array} \end{array}$$

Activity 6.1

1. Create a matrix from the following tables defining the rows and columns.

   (a) A sales representative had orders for the following types of garages for the first three months of the year.

	Jan	Feb	Mar
Domestic garage (6×6×2.4 m)	4	10	7
Domestic garage (9×6×2.4 m)	8	5	10
Machinery shed (18×9×3.6 m)	0	1	4

   (b) The following are prices in $ per night for motels at a coastal town.

	Luxury	Family	Single	Double
Sunny Resort	200	180	80	140
Sunny Surf	400	250	150	280
Shady Lane	180	180	50	90

2. Create a table for the number of hours you spend studying each of your subjects each week. Turn this table into a matrix with row and column headings.

6.1.2 Defining a matrix

Just like the convention we had in the Cartesian coordinate system whereby we defined a point by its *x* and *y* coordinates (in that order), we have a convention in the matrix system whereby we define each number (called an **element**) by its row and column. In the example of the page numbers,

$$\begin{array}{rc} & \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{array}{r} \text{Intro Book} \\ \text{Study Book 1} \\ \text{Study Book 2} \end{array} & \begin{pmatrix} 72 & 75 & 104 & 62 \\ 380 & 359 & 320 & 350 \\ 320 & 308 & 226 & 280 \end{pmatrix} \end{array}$$

Row 2,
Column 3

$320_{2,3}$ is the number of pages in study book 1 for level C. So this element of the matrix contains three pieces of information –

- The number of pages – 320
- The row number – 2
- The column number –3

If you look at the element $104_{1,3}$ in the matrix $\begin{pmatrix} 72 & 75 & 104 & 62 \\ 380 & 359 & 320 & 350 \\ 320 & 308 & 226 & 280 \end{pmatrix}$

this means there are 104 pages in row 1 of column 3, or in the introductory book of level C mathematics. The convention is to define the **row** first followed by the column.

In the Cartesian coordinate system, if we plot the point (3,8) we can uniquely identify the position of the point. This is similar in matrices. If we say show me the 3rd row of the 1st column, we can identify the value of this element. But we can symbolise this further. If we let any element in the matrix be *a*, then the 3rd row of the 1st column can be symbolised by $a_{3,1}$. In the matrix above the value of $a_{3,1}$ will be 320.

What is $a_{1,3}$? This is asking for the value of the 1st row of the 3rd column, which is 104 (there are 104 pages in the level C introductory book).

We can also define the size of the matrix by the number of rows and columns. In the mathematics pages example we say the matrix has **order of 3** × **4** (we say it is a 'three by four matrix'**).** This means the number of rows is 3 and the number of columns is 4. Look at the sweet company matrix again.

$$\begin{bmatrix} 150 & 120 & 200 & 100 & 300 & 350 \\ 100 & 100 & 150 & 50 & 120 & 200 \\ 80 & 50 & 50 & 100 & 50 & 100 \\ 50 & 40 & 80 & 100 & 50 & 80 \\ 100 & 80 & 200 & 200 & 150 & 300 \end{bmatrix}$$

The order of the matrix is 5 × 6 since there are 5 rows and 6 columns. Can you see again the number of rows is always stated first followed by the number of columns? While the symbol × does not strictly mean multiply, we may think of it this way when we want the total number of elements. In the first example on text books (a 3 × 4 matrix), there are 12 elements. In the second example on sweets (a 5 × 6 matrix), there are 30 elements.

6.1.3 Matrix equality

Two matrices are said to be equal if two conditions are satisfied:

- The two matrices are of the same order (i.e. they have the same number of rows and columns) and
- All the corresponding elements of the matrices are equal.

Example

Decide if the matrices below are equal.

$\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$ and $\begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}$

These are not equal since the elements in the first column have been reversed.

Example

Find values of the unknowns that make the matrices below equal.

$\begin{pmatrix} 1 & a \\ 7 & 7 \end{pmatrix}$ and $\begin{pmatrix} 1 & -4 \\ b & c \end{pmatrix}$

If $a = -4$, $b = 7$ and $c = 7$ then the matrices would be equal.

Activity 6.2

1. Define and give an example of each of the following:

	Definition	Example
(a) Matrix		
(b) Order of a matrix		
(c) Element of a matrix		
(d) $a_{i,j}$		
(e) Equal matrices		

2. The matrix below represents the probability of change in wheat prices with differences in weather conditions.

Change in wheat prices

$$\text{Weather conditions} \quad \begin{array}{r} \\ \text{dry} \\ \text{moderate} \\ \text{damp} \end{array} \begin{array}{c} \begin{array}{ccc} \text{up} & \text{down} & \text{no change} \end{array} \\ \begin{pmatrix} 0.6 & 0.1 & 0.3 \\ 0.4 & 0.2 & 0.4 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \end{array}$$

If $a_{3,3}$ means there is an 80% chance that the wheat prices will not change if the conditions are damp, what does $a_{2,1}$ mean?

3. A gold mining company has three sites. The Matilda, the Felicity and the Mt Granny, and has a plant that is able to extract gold from the ore.

   (a) The Matilda site had proved reserves of 500 000 tonnes, measured reserves of a million tonnes and inferred reserves of 200 000 tonnes. The Felicity mine had no proved reserves*, 70 000 tonnes of measured reserves and 23 000 tonnes of inferred reserves. The Mt Granny mine had no proved reserves, no measured reserves and 1.2 million tonnes of inferred reserves. From this information create a matrix with appropriate labels (express your answer as millions of tonnes).

   (b) The company has estimates of the number of grams per tonne it can extract from each reserve. From the Matilda site it is 1.8 g/t, 2.4 g/t and 2.8 g/t respectively; from the Felicity site it is 9.2 g/t and 2.4 g/t for the measured and inferred respectively and from the Mt Granny site it is 8.7 g/t for the inferred. Create a matrix showing this information with appropriate labels.

   * If there is no ore the matrix element is 0. All rows and columns in a matrix must be filled.

4. State the order of the following matrices:

   (a) $\begin{pmatrix} 2 & 0 & 0.8 \end{pmatrix}$

   (b) $\begin{pmatrix} 23 \\ 12 \\ 90 \\ 789 \\ 3 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 3 \\ 1 & 8 \\ 7 & 0 \\ 0 & 0.1 \\ 2 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 12 & -8 & \\ 3 & 4 & 1 \end{pmatrix}$

6.2 Calculating with matrices

As we have stated in the introduction, matrices have many applications both in solving equations and in geometrical transformations. We manipulate matrices just like we do any other numbers. But when we put numbers into matrix form they don't always obey the same rules as ordinary numbers. In the following section we will revise some of the rules of our number system and see how these rules apply to matrices.

6.2.1 Addition

In activity 6.1(a) a sales representative had orders for three types of garages for three months of the year. This can be described by the matrix:

$$\begin{array}{r c} & \begin{array}{ccc} \text{J} & \text{F} & \text{M} \end{array} \\ \begin{array}{r} \text{small} \\ \text{medium} \\ \text{mach. shed} \end{array} & \begin{pmatrix} 4 & 10 & 7 \\ 8 & 5 & 10 \\ 0 & 1 & 4 \end{pmatrix} \end{array}$$

If another sales representative had orders for the same types of garages for the same time period with the sales depicted as the matrix:

$$\begin{array}{r c} & \begin{array}{ccc} \text{J} & \text{F} & \text{M} \end{array} \\ \begin{array}{r} \text{small} \\ \text{medium} \\ \text{mach. shed} \end{array} & \begin{pmatrix} 0 & 4 & 8 \\ 0 & 3 & 10 \\ 0 & 1 & 14 \end{pmatrix} \end{array}$$

What are the total orders for each of the garages in each of the three months? In January the first salesman sold 4 of the small garages while the second salesman didn't sell any. So the element in the first row and first column would be 4. If we continue in this manner, we add each element of the first matrix with the corresponding element of the second matrix:

$$\begin{pmatrix} 4 & 10 & 7 \\ 8 & 5 & 10 \\ 0 & 1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 4 & 8 \\ 0 & 3 & 10 \\ 0 & 1 & 14 \end{pmatrix} = \begin{pmatrix} 4+0 & 10+4 & 7+8 \\ 8+0 & 5+3 & 10+10 \\ 0+0 & 1+1 & 4+14 \end{pmatrix} = \begin{pmatrix} 4 & 14 & 15 \\ 8 & 8 & 20 \\ 0 & 2 & 18 \end{pmatrix}$$

So to add matrices together, we simply add the corresponding elements. The matrices must be of the same order. We could not add the sales figures of another salesperson in the June, July period if he was selling 5 types of garages (this would be a 5×2 matrix). In practical situations even if the matrices were of the same order, sometimes it would not make sense if we added them. If we had a 3×3 matrix which described the colour and size of reams of paper, it would not make sense to add it to our sales representative's figures.

We can identify matrices by a capital letter. For the above example, if

$$A = \begin{pmatrix} 4 & 10 & 7 \\ 8 & 5 & 10 \\ 0 & 1 & 4 \end{pmatrix}, B = \begin{pmatrix} 0 & 4 & 8 \\ 0 & 3 & 10 \\ 0 & 1 & 14 \end{pmatrix} \text{ and } C = \begin{pmatrix} 4 & 14 & 15 \\ 8 & 8 & 20 \\ 0 & 2 & 18 \end{pmatrix}$$

we can say $A + B = C$

Activity 6.3

1. Find the sum of the following matrices (if possible).

(a) $A = \begin{pmatrix} 2 & -8 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 4 \\ 1 & 4 \end{pmatrix}$

(b) $X = \begin{pmatrix} 0.3 & 1.4 & 0.7 \end{pmatrix}, \quad Z = \begin{pmatrix} 2 & 0.7 \end{pmatrix}$

(c) $W = \begin{pmatrix} 2 & -8 & -3 \\ 1 & -9 & 0 \\ 0 & \frac{1}{4} & \frac{9}{10} \end{pmatrix}, \quad H = \begin{pmatrix} \frac{3}{4} & -1 & 9 \\ -1 & 2 & 0 \\ 2 & \frac{3}{4} & \frac{1}{2} \end{pmatrix}$

(d) $L = \begin{pmatrix} 2 & 1 \\ 6 & 7 \\ 1 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & -1 \\ -6 & -7 \\ -1 & -9 \end{pmatrix}$

(e) $S = \begin{pmatrix} 2 & 6 \\ 2 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 6 & 3 \\ 0 & 5 \end{pmatrix}, \quad T = \begin{pmatrix} 4 & 9 \\ 7 & 0 \end{pmatrix}$

(f) $V = \begin{pmatrix} x & y \\ x & 2y \end{pmatrix}, \quad G = \begin{pmatrix} 2 & 2y \\ x & 1 \end{pmatrix}$

2. A hairdresser owns two shops. Each shop sells shampoo, conditioner and styling gel in two brands 'own brand' and 'your style'. The shop in the shopping centre holds the following stock:

   'own brand': shampoo – 20; conditioner – 15; styling gel – 15

   'your style': shampoo – 5; conditioner – 5; styling gel – 8

   The shop in the suburbs holds the following stock:

   'own brand': shampoo – 10; conditioner – 5; styling gel – 5

   'your style': shampoo – 10; conditioner – 12; styling gel – 14

(a) Express the above information as two matrices one for the shopping centre and one for the suburban shop.

(b) What are the orders of the two matrices?

(c) Add these two matrices. What does the answer tell you in 'real life'?

Something to talk about...

There are an infinite number of examples where information can be expressed as a matrix. Create a realistic example yourself and talk about it with your colleagues or members of the discussion group.

6.2.2 Subtraction

The method of subtraction follows the same rules as those for addition. Let's look at another example.

A field biologist noted three types of birds in two different fields. He defined a 2 × 3 matrix:

$$\begin{array}{c} \\ \text{Field 1} \\ \text{Field 2} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Quail} & \text{Pheasant Caucal} & \text{Brush Turkey} \end{array} \\ \begin{pmatrix} 140 & 25 & 33 \\ 54 & 45 & 12 \end{pmatrix} \end{array}$$

Six months later, the quail population had decreased. The matrix below shows the number of birds that had disappeared from the fields.

$$\begin{pmatrix} 5 & 0 & 0 \\ 25 & 2 & 8 \end{pmatrix}$$

What are the new populations of birds in the two fields?

To find the population we would subtract the second matrix from the first matrix.

$$\begin{pmatrix} 140 & 25 & 33 \\ 54 & 45 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 25 & 2 & 8 \end{pmatrix} = \begin{pmatrix} 140-5 & 25-0 & 33-0 \\ 54-25 & 45-2 & 12-8 \end{pmatrix} = \begin{pmatrix} 135 & 25 & 33 \\ 29 & 43 & 4 \end{pmatrix}$$

So now there are 135 quail in field 1; 29 quail in field 2 etc.

Activity 6.4

1. Find the difference between the two matrices in activity 6.3 number 1(a).
2. Consider the matrices A and B in activity 6.3 number 1(a).

   (i) Prove $A + B = B + A$

   (ii) Prove $A - B \neq B - A$

Let's summarise what we know about the addition and subtraction of two matrices.

To add or subtract two matrices we must:

- make sure the number of rows and columns are the same for all matrices (i.e. the orders are the same)
- add or subtract each of the corresponding elements
- if the matrix operation comes from a real world problem, make sure it makes sense to add or subtract the matrices.

6.2.3 Multiplication

Suppose we had a knitting pattern that had either plain or purl types of stitches and three types of rows:

$$\begin{array}{r c} & \begin{array}{cc} \text{plain} & \text{purl} \end{array} \\ \begin{array}{r} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{array} & \begin{pmatrix} 20 & 10 \\ 15 & 15 \\ 10 & 20 \end{pmatrix} \end{array}$$

If we wanted to double the number of stitches in each row keeping the pattern the same, then the matrix would look like this:

$$\begin{pmatrix} 40 & 20 \\ 30 & 30 \\ 20 & 40 \end{pmatrix}$$

Another way of showing this would be:

$$2 \times \begin{pmatrix} 20 & 10 \\ 15 & 15 \\ 10 & 20 \end{pmatrix}$$

You can see that each element of the matrix is multiplied by 2. In this case we are multiplying a matrix by a number.

Example

If, in a particular year, the population of birds in a study depicted by the matrix below decreased by 20%. What is the new population of the birds?

$$\begin{array}{c} \\ \text{Field 1} \\ \text{Field 2} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Quail} & \text{Caucal} & \text{Turkey} \end{array} \\ \begin{pmatrix} 140 & 25 & 33 \\ 54 & 45 & 12 \end{pmatrix} \end{array}$$

Here we would multiply the matrix by 0.8 (the population decreased by 20%, so the remaining bird population would be 80%)

$$0.8 \times \begin{pmatrix} 140 & 25 & 33 \\ 54 & 45 & 12 \end{pmatrix} = \begin{pmatrix} 0.8\times 140 & 0.8\times 25 & 0.8\times 33 \\ 0.8\times 54 & 0.8\times 45 & 0.8\times 12 \end{pmatrix} = \begin{pmatrix} 112 & 20 & 26 \\ 43 & 36 & 10 \end{pmatrix}$$

(rounded to the nearest bird)

So now there are 112 quail in field 1, 43 quail in field 2 etc.

Activity 6.5

1. If $A = \begin{pmatrix} 2 & -8 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 4 \\ 1 & 4 \end{pmatrix}$ find:

   (a) $2 \times A$

   (b) $-3 \times B$

   (c) $-B$

2. In activity 6.2 the gold production company extraction figures were depicted as in the matrix below:

   $$\begin{pmatrix} 1.8 & 2.4 & 2.8 \\ 0 & 9.2 & 2.4 \\ 0 & 0 & 8.7 \end{pmatrix}$$

   Suppose the gold production company built a new processing plant that could extract 4% more ore than the old plant. Write this as a multiplication of a number by a matrix then write a matrix for the new extraction figures.

3. Express the following matrix as the product of a number and a matrix.

   $$\begin{pmatrix} \frac{1}{3} & \frac{11}{3} \\ \frac{2}{3} & \frac{-2}{3} \end{pmatrix}$$

The multiplication explained above was only a number multiplied by a matrix which is relatively straightforward. In the decimal number system we can also multiply numbers together. It is a bit more complicated than addition, but you have learnt a method (or algorithm) to perform the multiplication. In the following section you will learn how to multiply two matrices together.

Go back to the mathematics books example.

$$\begin{array}{r c}
 & \text{Level of maths} \\
 & \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\
\begin{array}{r} \text{Introductory Book} \\ \text{Study Book 1} \\ \text{Study Book 2} \end{array} & \begin{pmatrix} 72 & 75 & 104 & 62 \\ 380 & 359 & 320 & 350 \\ 320 & 308 & 226 & 280 \end{pmatrix}
\end{array}$$

Suppose we are only interested in materials for level B mathematics. This can be expressed as the column matrix $\begin{pmatrix} 75 \\ 359 \\ 308 \end{pmatrix}$.

We needed to do a print run. The printing company already had some stock of study book 1 and study book 2, but none of the introductory book. We put an order in for 200 of the introductory book, 50 of study book 1 and 50 of study book 2. We can express this order as a matrix. This time we will express it as a matrix with one row only $\begin{pmatrix} 200 & 50 & 50 \end{pmatrix}$. If the printers wanted to know how much paper they needed, they would have to do the following calculations:

200×75 this will be for the introductory book

50×359 this will be for study book 1

50×308 this will be for study book 2,

and then add the result. So the total number of pages to be printed would be $200 \times 75 + 50 \times 359 + 50 \times 308 = 48\,350$.

We can express this calculation in the form of a matrix multiplication:

$$\begin{pmatrix} 200 & 50 & 50 \end{pmatrix} \begin{pmatrix} 75 \\ 359 \\ 308 \end{pmatrix}$$

$$\begin{aligned} &= \begin{pmatrix} 200 \times 75 + & 50 \times 359 + & 50 \times 308 \end{pmatrix} \\ &= \begin{pmatrix} 15000 & + & 17950 & + & 15400 \end{pmatrix} \\ &= \begin{pmatrix} 48350 \end{pmatrix} \end{aligned}$$

Notice what is happening here. Column 1 of the first matrix (200) is multiplied by row 1 of the second matrix (75). Column 2 of the first matrix is multiplied by row 2 of the second matrix and column 3 of the first matrix is multiplied by row 3 of the second matrix.

$$\begin{pmatrix} 200 & 50 & 50 \end{pmatrix} \begin{pmatrix} 75 \\ 359 \\ 308 \end{pmatrix}$$

Example

Suppose we wanted to plant 8 gum trees and 4 grafted wattle trees. We could express this as a matrix:

$$\begin{array}{ccc} & & \text{tree} \\ \text{Type of tree} & \begin{array}{r} \text{gum} \\ \text{wattle} \end{array} & \begin{pmatrix} 8 \\ 4 \end{pmatrix} \end{array}$$

Now if we went to buy these trees and the cost was \$5 for a gum and \$15 for a grafted wattle, we could depict this as a 1×2 matrix.

$$\begin{array}{cc} & \text{Type of tree} \\ & \begin{array}{cc} \text{gum} & \text{wattle} \end{array} \\ \text{cost} & \begin{pmatrix} 5 & 15 \end{pmatrix} \end{array}$$

To find the total cost of the plants we can multiply these two matrices together.

$$\begin{pmatrix} 5 & 15 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \times 8 + 15 \times 4 \end{pmatrix} = \begin{pmatrix} 40 + 60 \end{pmatrix} = \begin{pmatrix} 100 \end{pmatrix}$$

Notice again the number of rows in the first matrix matches the number of columns in the second matrix.

Activity 6.6

1. Perform the following matrix multiplications (if possible).

   (a) $\begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 1 & 5 \end{pmatrix} \begin{pmatrix} -9 \\ -3 \\ 4 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$

In the maths book order example, if we wanted the same number of all four levels of books we could depict this as the matrix multiplication:

$$\begin{pmatrix} 200 & 50 & 50 \end{pmatrix}\begin{pmatrix} 72 & 75 & 104 & 62 \\ 380 & 359 & 320 & 350 \\ 320 & 308 & 226 & 280 \end{pmatrix}$$

Let's examine this matrix. In this case we are multiplying a 1×3 matrix by a 3×4 matrix. The labels on the columns of the first matrix are the types of book. This is the same as the labels of the rows of the second matrix.

Intro Book Study Book 1 Study Book 2

$$\begin{pmatrix} 200 & 50 & 50 \end{pmatrix} \quad \begin{pmatrix} 72 & 75 & 104 & 62 \\ 380 & 359 & 320 & 350 \\ 320 & 308 & 226 & 280 \end{pmatrix}$$

The columns of the first matrix **must** match the rows of the second matrix. If it doesn't then we can't perform the multiplication. It would be like putting in an order for an introductory book, a study book 1 and a study book 2 but having four types of maths books for each level to match it with.

Now we can perform the multiplication. Let's do one column at a time.

COLUMN 1

Each element of the row in the first matrix is multiplied by each element of column 1 in the second matrix.

$$\begin{pmatrix} 200 & 50 & 50 \end{pmatrix}\begin{pmatrix} 72 & 75 & 104 & 62 \\ 380 & 359 & 320 & 350 \\ 320 & 308 & 226 & 280 \end{pmatrix} \begin{aligned} &= (200 \times 72 + 50 \times 380 + 50 \times 320) \\ &= 1400 + 19000 + 16000 \\ &= (49400) \end{aligned}$$

In this case the printers would need 49 400 pages for level A mathematics.

COLUMN 2

$$\begin{pmatrix} 200 & 50 & 50 \end{pmatrix}\begin{pmatrix} 72 & 75 & 104 & 62 \\ 380 & 359 & 320 & 350 \\ 320 & 308 & 226 & 280 \end{pmatrix} = (200 \times 75 + 50 \times 359 + 50 \times 308)$$

$$= (15000 + 17950 + 15400)$$
$$= (48350)$$

In this case the printers would need 48 350 pages for level B mathematics.

COLUMN 3

$$\begin{pmatrix}200 & 50 & 50\end{pmatrix}\begin{pmatrix}72 & 75 & 104 & 62\\380 & 359 & 320 & 350\\320 & 308 & 226 & 280\end{pmatrix} = (200\times104+50\times320+50\times226)$$

$$= (20800+16000+11300)$$

$$= (48100)$$

In this case the printers would need 48 100 pages for level C mathematics.

COLUMN 4

$$\begin{pmatrix}200 & 50 & 50\end{pmatrix}\begin{pmatrix}72 & 75 & 104 & 62\\380 & 359 & 320 & 350\\320 & 308 & 226 & 280\end{pmatrix} = (200\times62+50\times350+50\times280)$$

$$= (12400+17500+14000)$$

$$= (43900)$$

In this case the printers would need 43 900 pages for level D mathematics.

So now we have 4 sets of figures, which we can put into matrix form.

This matrix gives us the total pages for the 4 levels of maths. Here is the matrix multiplication:

$$\begin{pmatrix}200 & 50 & 50\end{pmatrix}\begin{pmatrix}72 & 75 & 104 & 62\\380 & 359 & 320 & 350\\320 & 308 & 226 & 280\end{pmatrix} = \begin{pmatrix}49400 & 48350 & 48100 & 43900\end{pmatrix}$$

We started with a matrix of order 1×3 and multiplied it by a matrix of order 3×4. We now have a matrix of order 1×4. This also happened with the tree example. We started with a matrix of order 1×2 and multiplied it by a matrix of order 2×1. We then had a matrix of order 1×1.

$$\begin{pmatrix}200 & 50 & 50\end{pmatrix}\begin{pmatrix}72 & 75 & 104 & 62\\380 & 359 & 320 & 350\\320 & 308 & 226 & 280\end{pmatrix} = \begin{pmatrix}49400 & 48350 & 48100 & 43900\end{pmatrix}$$

Matrix: 1×3 3×4

Matrix: 1×4

$$\begin{pmatrix}5 & 15\end{pmatrix}\begin{pmatrix}8\\4\end{pmatrix} = (100)$$

Matrix: 1×2 2×1

Matrix: 1×1

When we want to multiply two matrices together, we must do two things before we do the multiplication:

- See if the number of columns of the first matrix must match the number of rows of the second matrix.
- Decide on the order of the resulting matrix. The resulting matrix will have the number of rows in the first matrix and the number of columns in the second matrix.

If you write the two matrix orders side by side, you can see this more clearly.

Example

Consider the matrix multiplication

$$\begin{pmatrix} 5 & 15 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

- Column and row the same? Yes

$$\boxed{\text{Matrices: } 1 \times 2 \quad 2 \times 1}$$

- New matrix order

$$\boxed{\text{Matrix } 1 \times 1}$$

Example

Consider the matrix

$$\begin{pmatrix} 200 & 50 & 50 \end{pmatrix} \begin{pmatrix} 72 & 75 & 104 & 62 \\ 380 & 359 & 320 & 350 \\ 320 & 308 & 226 & 280 \end{pmatrix}$$

- Column and row the same? Yes

$$\boxed{\text{Matrices: } 1 \times 3 \quad 3 \times 4}$$

- New matrix order

$$\boxed{\text{Matrix } 1 \times 4}$$

Example

Look at the following matrix multiplication and answer the following questions.

(a) What is the order of each of the matrices?

(b) What will be the order of the resulting matrix?

(c) Perform the multiplication.

$$\begin{pmatrix} 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 6 & 0 \\ 1 & 2 & 4 \end{pmatrix}$$

(a) orders of matrices are 1×2 and 2×3

(b) order of resulting matrix is 1×3

(c) $\begin{pmatrix} 4+4 & 12+8 & 0+16 \end{pmatrix}$
$= \begin{pmatrix} 8 & 20 & 16 \end{pmatrix}$

Example

Decide whether the following matrix multiplication can be performed. If so find the order of the new matrix and then perform the multiplication.

$$\begin{pmatrix} 9 & 1 & 0.3 \end{pmatrix}\begin{pmatrix} 1 & 0.2 & 5 \\ 2 & 3 & 2 \\ 1 & 4 & 0.8 \end{pmatrix}$$

Since the orders of matrices are 1×3 and 3×3 the number of columns of the first matrix is the same as the number of rows of the second matrix. The resulting matrix is of order 1×3

$$\begin{pmatrix} 9 & 1 & 0.3 \end{pmatrix}\begin{pmatrix} 1 & 0.2 & 5 \\ 2 & 3 & 2 \\ 1 & 4 & 0.8 \end{pmatrix} \begin{aligned} &= \begin{pmatrix} 9+2+0.3 & 1.8+3+1.2 & 45+2+0.24 \end{pmatrix} \\ &= \begin{pmatrix} 11.3 & 6 & 47.24 \end{pmatrix} \end{aligned}$$

Activity 6.7

1. Examine the following matrix multiplication.

$$\begin{pmatrix} 3 & 9 \end{pmatrix}\begin{pmatrix} 1 & 5 & 0 \\ 2 & -9 & 2 \end{pmatrix}$$

   (a) What is the order of each of the matrices?

   (b) What will be the order of the resulting matrix?

   (c) Perform the multiplication.

2. Decide whether the following matrix multiplication can be performed. If so find the order of the new matrix and then perform the multiplication.

$$\begin{pmatrix} 9 & 1 & 0.3 \end{pmatrix}\begin{pmatrix} 3 & 8 \\ 1 & 0 \\ 4 & 1 \end{pmatrix}$$

Suppose the Tooth Rot Sweet Company made three types of sweets on a particular machine – snakes, musks and bananas. Two of the ingredients they had in common were sugar and glucose. The percentages of these two ingredients are shown in the matrix below (expressed as decimals):

$$\begin{array}{ccc} & & \text{product} \\ & & \begin{array}{ccc}\text{snakes} & \text{musks} & \text{bananas}\end{array} \\ \text{ingredient} & \begin{array}{r}\text{sugar} \\ \text{glucose}\end{array} & \begin{pmatrix} 0.30 & 0.40 & 0.35 \\ 0.30 & 0.20 & 0.25 \end{pmatrix} \end{array}$$

The company has two factories that wish to order ingredients based on the number of kilograms of each sweet they produce. Below is the matrix of the number of kilograms of each type of sweet made in each factory.

$$\begin{array}{cc} & \text{Factory} \\ & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{r}\text{snakes} \\ \text{musks} \\ \text{bananas}\end{array} & \begin{pmatrix} 100 & 400 \\ 200 & 100 \\ 80 & 400 \end{pmatrix} \end{array}$$

In order to find the number of kilos of glucose and sugar, we must perform a matrix multiplication.

$$\begin{pmatrix} 0.30 & 0.40 & 0.35 \\ 0.30 & 0.20 & 0.25 \end{pmatrix}\begin{pmatrix} 100 & 400 \\ 200 & 100 \\ 80 & 400 \end{pmatrix}$$

If we multiply each element of the first row with each element of the first column, then we get the total amount of sugar needed for factory 1.

$$\begin{pmatrix} 0.30 & 0.40 & 0.35 \\ 0.30 & 0.20 & 0.25 \end{pmatrix}\begin{pmatrix} 100 & 400 \\ 200 & 100 \\ 80 & 400 \end{pmatrix} = 0.30\times 100 + 0.40\times 200 + 0.35\times 80 = 30 + 80 + 28 = 138$$

Now if we multiply each element of the first row with each element of the second column, then we get the total amount of sugar needed for the second factory.

$$\begin{aligned}\begin{pmatrix} 0.30 & 0.40 & 0.35 \\ 0.30 & 0.20 & 0.25 \end{pmatrix}\begin{pmatrix} 100 & 400 \\ 200 & 100 \\ 80 & 400 \end{pmatrix} &= 0.30\times 400 + 0.40\times 100 + 0.35\times 400 \\ &= 120 + 40 + 140 \\ &= 300\end{aligned}$$

If we now look at the second row of the first matrix – this is the glucose. If we performed the same matrix multiplication for the glucose for factory one and two, we would get the total glucose for the factories.

$$\begin{pmatrix} 0.30 & 0.40 & 0.35 \\ 0.30 & 0.20 & 0.25 \end{pmatrix}\begin{pmatrix} 100 & 400 \\ 200 & 100 \\ 80 & 400 \end{pmatrix} = 0.30\times 100 + 0.20\times 200 + 0.25\times 80 = 30 + 40 + 20 = 90$$

$$\begin{aligned}\begin{pmatrix} 0.30 & 0.40 & 0.35 \\ 0.30 & 0.20 & 0.25 \end{pmatrix}\begin{pmatrix} 100 & 400 \\ 200 & 100 \\ 80 & 400 \end{pmatrix} &= 0.30\times 400 + 0.20\times 100 + 0.25\times 400 = 120 + 20 + 100 \\ &= 240\end{aligned}$$

Now let's put this together.

$$\begin{pmatrix} 0.30 & 0.40 & 0.35 \\ 0.30 & 0.20 & 0.25 \end{pmatrix}\begin{pmatrix} 100 & 400 \\ 200 & 100 \\ 80 & 400 \end{pmatrix} = \begin{pmatrix} 138 & 300 \\ 90 & 240 \end{pmatrix}$$

The resulting 2×2 matrix shows the total amount of sugar and glucose needed for the two factories.

$$\begin{array}{ccc} & & \text{factory} \\ & & \begin{array}{cc} 1 & 2 \end{array} \\ \text{ingredients} & \begin{array}{r} \text{sugar} \\ \text{glucose} \end{array} & \begin{pmatrix} 138 & 300 \\ 90 & 240 \end{pmatrix} \end{array}$$

So we need 138 kilograms of sugar for factory 1 and 300 for factory 2. We need 90 kilograms of glucose for factory 1 and 240 for factory 2.

Notice we multiplied a 2×3 matrix by a 3×2 matrix, and the resulting matrix was a 2×2 matrix.

Example

1. Examine the following matrix multiplication

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 6 & 0 \\ 1 & 2 & 4 \end{pmatrix}$$

(a) What is the order of each of the matrices?

(b) What will be the order of the resulting matrix?

(c) Perform the multiplication.

(a) orders of matrices are 2×2 and 2×3

(b) order of resulting matrix is 2×3

(c) $$\begin{pmatrix} 2+3 & 6+6 & 0+12 \\ 4+4 & 12+8 & 0+16 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 12 & 12 \\ 8 & 20 & 16 \end{pmatrix}$$

Example

Decide whether the following matrix multiplication can be performed. If so find the order of the new matrix and then perform the multiplication.

$$\begin{pmatrix} 2 \\ 8 \end{pmatrix}\begin{pmatrix} 3 & 6 & 1 & 4 \end{pmatrix}$$

The order of the matrices are 2×1 and 1×4 so the multiplication can be performed since the number of columns of matrix 1 is equal to the number of rows of matrix two. The resulting matrix is of order 2×4.

$$\begin{pmatrix} 2 \\ 8 \end{pmatrix}\begin{pmatrix} 3 & 6 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 12 & 2 & 8 \\ 24 & 48 & 8 & 32 \end{pmatrix}$$

Activity 6.8

1. Examine the following matrix multiplication

$$\begin{pmatrix} 3 & 9 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 5 & 0 \\ 2 & -9 & 2 \end{pmatrix}$$

   (a) What is the order of each of the matrices?

   (b) What will be the order of the resulting matrix?

   (c) Perform the multiplication.

2. Decide whether the following matrix multiplication can be performed. If so find the order of the new matrix and then perform the multiplication.

$$\begin{pmatrix} 9 & 1 & 0.3 \\ 4 & 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & 8 \\ 1 & 0 \\ 4 & 1 \end{pmatrix}$$

3. If $A = \begin{pmatrix} 3 & 6 \\ 6 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 8 \\ 4 & 2 \end{pmatrix}$ find

   (a) $A \times B$

   (b) $B \times A$

   (c) Do you think the general statement $A \times B = B \times A$, about matrix multiplication would be true? Give reasons for your answer.

4. Find the product of the following pairs of matrices (in the order they appear) if possible

   (a) $A = \begin{pmatrix} 2 & -8 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 4 \\ 1 & 4 \end{pmatrix}$

   (b) $X = \begin{pmatrix} 0.3 & 1.4 & 0.7 \end{pmatrix}$, $Z = \begin{pmatrix} 2 & 0.7 \end{pmatrix}$

   (c) $W = \begin{pmatrix} 2 & -8 & -3 \\ 1 & -9 & 0 \\ 0 & \frac{1}{4} & \frac{9}{10} \end{pmatrix}$, $H = \begin{pmatrix} \frac{3}{4} & -1 & 9 \\ -1 & 2 & 0 \\ 2 & \frac{3}{4} & \frac{1}{2} \end{pmatrix}$

   (d) $L = \begin{pmatrix} 2 & 1 \\ 6 & 7 \\ 1 & 9 \end{pmatrix}$, $C = \begin{pmatrix} -2 & -1 \\ -6 & -7 \\ -1 & -9 \end{pmatrix}$

   (e) $S = \begin{pmatrix} 2 & 6 \\ 2 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 6 & 3 \\ 0 & 5 \end{pmatrix}$, $T = \begin{pmatrix} 4 & 9 \\ 7 & 0 \end{pmatrix}$

   (f) $V = \begin{pmatrix} x & y \\ x & 2y \end{pmatrix}$, $G = \begin{pmatrix} 2 & 2y \\ x & 1 \end{pmatrix}$

Something to talk about...

Are you finding the multiplication of matrices difficult? How can you remember the order of what to do? Perhaps you would like to share any tips with others in the discussion group.

6.3 Some special matrices

We have looked at how to add, subtract and multiply matrices. We saw that in many cases this was similar to the ordinary decimal system with a few restrictions. Let's continue with this comparison. We cannot really divide by a matrix, but we can look at some similar operations in the decimal number system related to division. These are the identity and the inverse.

6.3.1 Identity matrices

Look at the following example in the decimal number system.

$3 \times 1 = 3$

The number 1 is called the **identity element** of multiplication, since when you multiply 1 by any number it gives that number. (i.e. $x \times 1 = x$ or $1 \times x = x$).

Similarly in matrix multiplication there is an identity matrix, such that if you multiply it by any matrix the resulting matrix is the same matrix. Look at the following examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 3+0 & 1+0 \\ 0+4 & 0+5 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}$$

which is the same matrix we started with.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0.7 & 1 \\ -5 & 3 & -9 \\ 3 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1+0+0 & 0.7+0+0 & 1+0+0 \\ 0+-5+0 & 0+3+0 & 0+-9+0 \\ 0+0+3 & 0+0+0 & 0+0+4 \end{pmatrix} \begin{pmatrix} 1 & 0.7 & 1 \\ -5 & 3 & -9 \\ 3 & 0 & 4 \end{pmatrix}$$

Again, this is the same matrix we started with.

Look carefully at the matrix multiplication above. Can you answer these questions:

1. Does first matrix always have to be the identity matrix?
2. Does the identity matrix have to have the same number of rows and columns?
3. Does the diagonal of ones have to run down from left to right with the remaining elements 0?

Let's have a closer look at these points.

1. If we multiplied an identity matrix by a matrix would we still get the original matrix?

$$\begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3+0 & 0+1 \\ 4+0 & 0+5 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}$$

   Yes! Pre- or post- multiplying by the identity matrix does not change a square matrix. Try it yourself. You will see no matter what values we have in the original matrix, it will always work.

2. What if we had a matrix that was not square?

$$\begin{pmatrix} 2 & 3 \\ 4 & 7 \\ 1 & 2 \end{pmatrix}$$

   This is a 3×2 matrix. If we multiplied it by another matrix and we wanted the original matrix, what size would it have to be?

   It would have to be a 3×3 matrix since a 3×3 times a 3×2 give a 3×2 matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 7 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 7 \\ 1 & 2 \end{pmatrix}$$

   But what if we post multiplied by an identity matrix? Can you see a problem?

   Since the first matrix is a 3×2 matrix we need the second matrix to have order 2×2, so we can get a 3×2. So would an identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ work?

$$\begin{pmatrix} 2 & 3 \\ 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2+0 & 0+3 \\ 4+0 & 0+7 \\ 1+0 & 0+2 \end{pmatrix}$$

Most of the matrices you will be dealing with in relation to the identity will be square matrices, but as you can see you have to be careful when pre and post multiplying matrices.

But remember that an identity matrix is always a square matrix (i.e. the number of rows and columns are equal).

3. Try this multiplication:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 7 \\ 1 & 2 \end{pmatrix}$$

Did you get the original matrix?

No! You should have $\begin{pmatrix} 1 & 2 \\ 4 & 7 \\ 2 & 3 \end{pmatrix}$

An identity matrix must always have the 1's running from left to right.

Activity 6.9

1. Write an identity matrix for the following matrices:

(a) $\begin{pmatrix} 3 & 6 & 1 & 3 \\ 2 & 4 & 2 & 0 \\ 1 & 3 & 8 & 1 \\ 4 & 2 & 1 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 6 & -6 \\ 1 & 10 & 1 \end{pmatrix}$ (identify if you are pre- or post-multiplying)

2. Fill in the missing elements in the equation:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 9 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 3 & x \\ 9 & y \end{pmatrix}$$

6.3.2 The inverse matrix

Let's again visit the decimal number system. This time have a look at the equation:

$$3 \times \frac{1}{3} = 1$$

Any number multiplied by its reciprocal is equal to 1. In mathematics terms, we call $\frac{1}{3}$ the **multiplicative inverse** of 3. You have come across this idea of inverses in arithmetic, in functions, now it is also an important part of matrices.

An inverse matrix, when multiplied by the original matrix gives the identity matrix. Let's have a look at some examples.

Example

Is the inverse of $\begin{pmatrix} 3 & -4 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{7} & \frac{4}{7} \\ \frac{-1}{7} & \frac{3}{7} \end{pmatrix}$?

To show that the matrix is the inverse, we multiply them together. If the result is the identity matrix, then it is the inverse.

$$\begin{pmatrix} \frac{1}{7} & \frac{4}{7} \\ \frac{-1}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{7}+\frac{4}{7} & \frac{-4}{7}+\frac{4}{7} \\ \frac{-3}{7}+\frac{3}{7} & \frac{4}{7}+\frac{3}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or we could have multiplied them the other way:

$$\begin{pmatrix} 3 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{7} & \frac{4}{7} \\ \frac{-1}{7} & \frac{3}{7} \end{pmatrix} = \begin{pmatrix} \frac{3}{7}+\frac{4}{7} & \frac{-4}{7}+\frac{4}{7} \\ \frac{-3}{7}+\frac{3}{7} & \frac{4}{7}+\frac{3}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So the inverse of $\begin{pmatrix} 3 & -4 \\ 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{7} & \frac{4}{7} \\ \frac{-1}{7} & \frac{3}{7} \end{pmatrix}$.

Since they are square matrices, it doesn't matter if we pre- or post-multiply the inverse.

$\begin{pmatrix} 3 & -4 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{7} & \frac{4}{7} \\ \frac{-1}{7} & \frac{3}{7} \end{pmatrix}$ are inverses of each other.

In this level of mathematics you will always be given the inverse of the matrix. If you go on to do more mathematics, you may learn how to find the inverse of quite large matrices.

Important note: We cannot divide one matrix by another matrix, but we can divide a matrix by a constant, e.g.

$$\text{If} \quad M = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} \quad \text{then} \quad \frac{m}{2} = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 2 & \frac{7}{2} \end{bmatrix}$$

For interest only…

You may like to know how to find the inverse of a 2 × 2 matrix. You DO NOT have to learn this. If you had a matrix A = $\begin{pmatrix} \mathbf{1} & 3 \\ 7 & \mathbf{2} \end{pmatrix}$ then to find the inverse, you interchange the element 1 with 2 and make the 7 and 3 negative. So the matrix now becomes $\begin{pmatrix} 2 & -3 \\ -7 & 1 \end{pmatrix}$.
Next you multiply the whole matrix by 1 over the product of 1 and 2 minus the product of 7 and 3. The matrix is now

$$\frac{1}{2-21}\begin{pmatrix} 2 & -3 \\ -7 & 1 \end{pmatrix} = \frac{1}{-19}\begin{pmatrix} 2 & -3 \\ -7 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{19} & \frac{3}{19} \\ \frac{7}{19} & -\frac{1}{19} \end{pmatrix}$$

More generally the inverse of a 2 × 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Do not learn this.

Activity 6.10

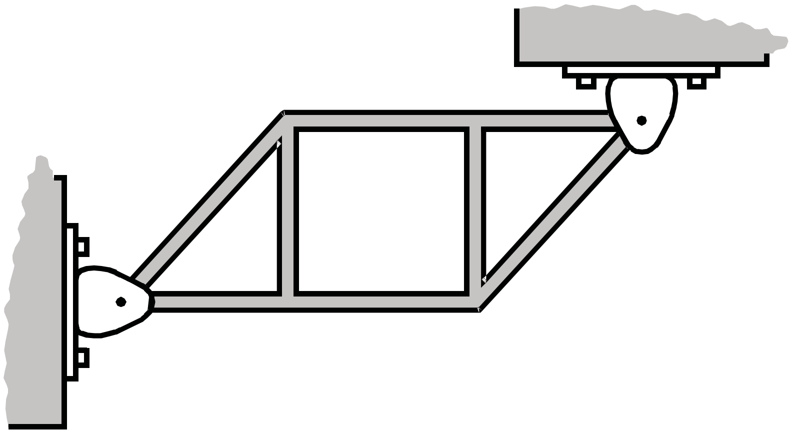
1. Show the matrices below are inverses of each other.

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

6.4 Solving matrix equations

One of the most important uses for identities and inverses is in solving equations.

At the beginning of this module we saw a problem that had 4 equations. Often we may come across larger problems. For example look at the truss below.



We are not going to study forces but for every joint in the truss two equations are generated, which gives us a total of 12 simultaneous equations with 12 variables. This information can be put into a 12 × 12 matrix.

$$\begin{pmatrix}
1 & 0 & \frac{-1}{\sqrt{2}} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{\sqrt{2}} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 1
\end{pmatrix}$$

Luckily in this unit you don't have to manipulate 12 × 12 matrices! But we will look at how a matrix can help us solve simultaneous equations.

First let us see how we can convert simultaneous equations into a matrix form.

If we had the matrix multiplication:

$$\begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

we would then get the matrix

$$\begin{pmatrix} 2x+5y \\ 3x+7y \end{pmatrix}$$

if we made this matrix equal to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

then we would have the equation

$$\begin{pmatrix} 2x+5y \\ 3x+7y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

which is the same as saying:

$2x + 5y = 3$ and

$3x + 7y = 4$

Look at the simultaneous equations above and the equivalent matrix equation below. Can you see how to convert from the algebraic to the matrix form?

$$\begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Let's look at a similar example

$$5x + -2y = 0$$
$$2x + 1y = 11$$

In this case we have to make each term connected by a + sign and make sure each variable has a coefficient.

$$5x + -2y = 0$$
$$2x + 1y = 11$$

Now the matrix equation becomes

$$\begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 11 \end{pmatrix}$$

Look at the example posed at the beginning of the module. Let's convert it into matrix form. Here Y = income; C = consumption; I = investment and R = interest rate.

$$1Y - 1C - 1I + 0R = \overline{G}$$
$$-bY + 1C + 0I + 0R = a$$
$$-dY + 0C + 1I - eR = 0$$
$$1Y + 0C + 0I + \frac{g}{f} \times R = \frac{1}{f} \times \overline{M}$$

To convert it to a matrix form we must follow a few steps.

- First we must identify the unknowns. In this case they are Y C I and R.
- Now we must identify the coefficients of the unknowns (they are highlighted in bold) and remember to include negatives where necessary. Notice here some of the coefficients are not numerical values.
- Finally create three matrices.

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -b & 1 & 0 & 0 \\ -d & 0 & 1 & -e \\ 1 & 0 & 0 & \frac{g}{f} \end{pmatrix}\begin{pmatrix} Y \\ C \\ I \\ R \end{pmatrix}=\begin{pmatrix} \overline{G} \\ a \\ 0 \\ \frac{1}{f}\overline{M} \end{pmatrix}$$

In this level of mathematics we will not ask you to solve 4 × 4 matrix equations, because it is quite time consuming, but you should be able to see the process is the same no matter how big the matrix.

To summarise we could say:

- Make sure each equation is in a standard form Ax + By +… = C with the constant on the right hand side and the terms on the left hand side separated by a + sign.
- Identify the coefficients of the unknowns.
- Each of the coefficients of the unknowns form the elements of the first matrix. This matrix is sometimes called the **coefficient matrix**.
- This first matrix is multiplied by a single column matrix with the elements being the unknowns e.g. $\begin{pmatrix} x \\ y \end{pmatrix}$
- The constants in the equations become the elements of the third matrix.

Example

Convert the following simultaneous equations into a matrix equation.

$$5x + 2y = 4$$
$$7x - y = -3$$

In this example the equations are already in the correct form with the unknowns being *x* and *y*. We need to identify the coefficients – they are shown in bold below.

$$\mathbf{5}x + \mathbf{2}y = 4$$
$$\mathbf{7}x + \mathbf{-1}y = -3$$

Now we create three matrices.

$$\begin{pmatrix} 5 & 2 \\ 7 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

Example

Write a single matrix equation from these simultaneous equations.

$$y = 3x - 7$$
$$x + 2y = 1$$

We must rearrange the first equation into the standard form $Ax + By = C$ and use + between terms.

$$3x + -y = 7$$

So the equations now become:

$$3x + -y = 7$$
$$x + 2y = 1$$

Make sure there is a coefficient next to each unknown on the left hand side.

$$3x + -1y = 7 \text{ - (i)}$$
$$1x + 2y = 1 \text{ - (ii)}$$

Now put equations (i) and (ii) in matrix form.

$$3x + -1y = 7$$
$$1x + 2y = 10$$
$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

Example

Convert the following into a matrix equation.

$$\begin{aligned} 7x + 2y + z &= 10 \\ x + y &= 2z \\ 2x - 3y &= 0 \end{aligned}$$

First put it into a standard form with + signs between the terms.

$$\begin{aligned} 7x + 2y + z &= 10 \\ x + y + -2z &= 0 \\ 2x + -3y &= 0 \end{aligned}$$

Now make sure there are coefficients for each variable.

$$\begin{aligned} 7x + 2y + 1z &= 10 \\ 1x + 1y + -2z &= 0 \\ 2x + -3y + 0z &= 0 \end{aligned}$$

Now create the three matrices.

$$\begin{pmatrix} 7 & 2 & 1 \\ 1 & 1 & -2 \\ 2 & -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

Activity 6.11

1. Change the following simultaneous equations into matrix form.

   (a) $$\begin{aligned} 7x + 2y &= -1 \\ x + 4y &= 9 \end{aligned}$$

   (b) $$\begin{aligned} y &= 2x + 4 \\ 5y + 3 &= 2x \end{aligned}$$

   (c) $$\begin{aligned} y &= 3x + 2z \\ 4x + y &= 12 \\ 3x + 9y + 4z &= 9 \end{aligned}$$

Before we try to solve these matrix equations let's recap one important point when solving ordinary equations.

If we had an equation: $3y = 21$

To find a value for y we would multiply both sides by $\frac{1}{3}$. So we have:

$$\frac{1}{3} \times 3y = \frac{1}{3} \times 21$$
$$1 \times y = 7$$

In this case the $\frac{1}{3}$ is called the **multiplicative inverse**. We use the same process with matrix equations. However, while in ordinary arithmetic it doesn't matter in which order we place the factors 3, $\frac{1}{3}$ and y, in matrix algebra the position of the matrices is important.

So if we had a set of equations we would first convert them into matrix form.

$3x + 4y = 10$

$2x + 5y = 2$

Change this into a matrix equation:

$$\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

We want to get the unknown matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ by itself. Just like ordinary solving of algebraic equations, we must multiply both sides by the multiplicative inverse which in this case is $\begin{pmatrix} \frac{5}{7} & \frac{-4}{7} \\ \frac{-2}{7} & \frac{3}{7} \end{pmatrix}$.

$$\begin{pmatrix} \frac{5}{7} & \frac{-4}{7} \\ \frac{-2}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5}{7} & \frac{-4}{7} \\ \frac{-2}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

The first two matrices multiply to give the identity matrix (check this carefully yourself). We then multiply the last two matrices to get the solution to our equation. Remember we have to pre-multiply by the inverse on both the left- and right-hand sides otherwise we will not get the correct answer.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{50}{7} + \frac{-8}{7} \\ \frac{-20}{7} + \frac{6}{7} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{42}{7} \\ \frac{-14}{7} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

So $x = 6$ and $y = -2$

Check in both equations

LHS = 3 × 6 + 4 × –2 = 10 = RHS

LHS = 2 × 6 + 5 × –2 = 2 = RHS

Let's have a look at the steps involved in the following examples.

Example

Using matrices solve the simultaneous equations

$$\begin{aligned} x &= 2y \\ 2x &= y + 4 \end{aligned}$$

given the inverse of the matrix $\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ is $\begin{pmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$

Step 1: Rearrange into a standard form	$x = 2y \quad \Rightarrow 1x - 2y = 0$ $2x = y + 4 \Rightarrow 2x - 1y = 4$
Step 2: Put new equations into matrix form	$\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$
Step 3: Pre-multiply both sides by the inverse matrix	$\begin{pmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{4}{3} \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{4}{3} \end{pmatrix}$
Step 4: State the solution	Therefore $x = \frac{8}{3}$ and $y = \frac{4}{3}$
Step 5: Check in the original equations	$x = 2y \Rightarrow \text{RHS} = 2 \times \frac{4}{3} = \frac{8}{3} = \text{LHS}$ $2x = y + 4 \Rightarrow \text{LHS} = 2 \times \frac{8}{3} = \frac{16}{3}$ $\text{RHS} = \frac{4}{3} + 4 = \frac{16}{3} = \text{LHS}$

Example

If A is $\begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$ and B is $\begin{pmatrix} -5 & -3 \\ 6 & 4 \end{pmatrix}$, solve the matrix equation $A = BX$. (The inverse of A $= \begin{pmatrix} \frac{1}{14} & \frac{-2}{7} \\ \frac{3}{14} & \frac{1}{7} \end{pmatrix}$ and the inverse of B $= \begin{pmatrix} -2 & \frac{-3}{2} \\ 3 & \frac{5}{2} \end{pmatrix}$)

To solve $A = BX$, we must have X by itself, so we must pre-multiply both sides by the inverse of B.

$$\begin{pmatrix} -2 & \frac{-3}{2} \\ 3 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix} = X$$

$$= \begin{pmatrix} -4 + \frac{9}{2} & -8 - \frac{3}{2} \\ 6 - \frac{15}{2} & 12 + \frac{5}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{2} & \frac{-19}{2} \\ \frac{-3}{2} & \frac{29}{2} \end{pmatrix} \text{ or we could write it as } \frac{1}{2} \times \begin{pmatrix} 1 & 19 \\ -3 & 29 \end{pmatrix}$$

Example

Solve the following simultaneous equations:

$$2x + 3y - 2z = 5$$
$$2x + y + z = 11$$
$$3x + 2y - 3z = 0$$

given that the inverse of $\begin{pmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -3 \end{pmatrix}$ is $\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{5} & 0 & \frac{-2}{5} \\ \frac{1}{15} & \frac{1}{3} & \frac{-4}{15} \end{pmatrix}$

In matrix form the above equation becomes

$$\begin{pmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{5} & 0 & \frac{-2}{5} \\ \frac{1}{15} & \frac{1}{3} & \frac{-4}{15} \end{pmatrix}\begin{pmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{5} & 0 & \frac{-2}{5} \\ \frac{1}{15} & \frac{1}{3} & \frac{-4}{15} \end{pmatrix}\begin{pmatrix} 5 \\ 11 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-5}{3} + \frac{11}{3} + 0 \\ 3 + 0 + 0 \\ \frac{1}{3} + \frac{11}{3} + 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

So the solution to the given simultaneous equations is: $x = 2$, $y = 3$, and $z = 4$.

You should check these solutions in each of the 3 original equations.

Activity 6.12

1. Write the following simultaneous equations in matrix form and then solve for x and y using matrix multiplication:

   $$2x + y = 5$$
   $$x + 2y = 4$$

   Hint: The inverse of the coefficient matrix is: $\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

2. Solve the simultaneous equations

   $$x + 2y + 3z = 2$$
   $$2x + 5y + 3z = 6$$
   $$x + 8z = 1$$

   using matrices given the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$ is $\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$

3. If $C=\begin{pmatrix}-8 & 4\\ -16 & 4\end{pmatrix}$ and $M=\begin{pmatrix}7 & 3\\ 4 & 2\end{pmatrix}$

and the inverses of C and M are $\begin{pmatrix}\frac{1}{8} & -\frac{1}{8}\\ \frac{1}{2} & -\frac{1}{4}\end{pmatrix}$ and $\begin{pmatrix}1 & -\frac{3}{2}\\ -2 & \frac{7}{2}\end{pmatrix}$ respectively,

solve for P in the equation $CP = M$

Something to talk about...

In activity 6.11 number 1(b) the solution we had was $\begin{pmatrix}-2 & 1\\ -2 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}4\\ -3\end{pmatrix}$

We could have rearranged the equations to get $\begin{aligned}2x-y&=-4\\ 2x-5y&=3\end{aligned}$

To give the matrix equation

$$\begin{pmatrix}2 & -1\\ 2 & -5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-4\\ 3\end{pmatrix}$$

Look at the various solutions you could have and discuss with colleagues or the discussion group why these are the same.

6.5 Real world problems

During this module you have come across some examples of matrices in everyday situations. Let's have a closer look at matrices using some 'case studies' from business and graphic art. You may have to draw on your knowledge of algebra and graphing to do these examples.

Example

Supply and demand are important concepts in economics. These are often expressed in terms of the price people are willing to pay for goods and the price dealers are willing to sell these goods for. Suppose we are given a supply and demand market model.

Demand function: $1p = 40 - 0.40q$

Supply function: $1p = -2.5 + 0.25q$

(p = price of the commodity in dollars and q is the quantity in units). Equilibrium occurs when there are values of p and q that satisfies both equations.

Suppose the imposition of a tax changes the constant in the supply function from –2.5 to $(t - 2.5)$.

To include the tax changes the supply equation now becomes:

Supply function : $1p = (t - 2.5) + 0.25q$

To solve the two simultaneous equations we can use matrices. Note we will have t as a variable in the solution.

We must first change to a standard form:

$$1p + 0.40q = 40$$
$$1p + -0.25q = (t - 2.5)$$

The matrix equation now becomes

$$\begin{pmatrix} 1 & 0.40 \\ 1 & -0.25 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 40 \\ t - 2.5 \end{pmatrix}$$

To solve this equation we need the inverse of $\begin{pmatrix} 1 & 0.40 \\ 1 & -\frac{1}{4} \end{pmatrix}$ which is $\begin{pmatrix} \frac{5}{13} & \frac{8}{13} \\ \frac{20}{13} & -\frac{20}{13} \end{pmatrix}$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{5}{13} & \frac{8}{13} \\ \frac{20}{13} & -\frac{20}{13} \end{pmatrix}\begin{pmatrix} 40 \\ t - 2.5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{200}{13} + \frac{8(t - 2.5)}{13} \\ \frac{800}{13} + \frac{-20(t - 2.5)}{13} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{200 + 8t - 20}{13} \\ \frac{800 - 20t + 50}{13} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8t + 180}{13} \\ \frac{-20t + 850}{13} \end{pmatrix}$$

or

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{8t}{13} + \frac{180}{13} \\ \frac{-20t}{13} + \frac{850}{13} \end{pmatrix}$$

If there was no tax, then $t = 0$

$p = \dfrac{8 \times 0 + 180}{13}$ and $q = \dfrac{-20 \times 0 + 850}{13}$

the equilibrium price would be about \$13.85, the quantity would be about 65.385 units

If a tax of \$1 was imposed the new equilibrium price would be

$p = \dfrac{8 \times 1 + 180}{13} = \$14.46$ and the new quantity would be

$q = \dfrac{-20 \times 1 + 850}{13} = 63.846$ units

Look at the following sentence. Complete the statement:

> *For each unit of tax that is imposed the equilibrium price rises/falls by___________ while the equilibrium quantity rises/falls by __________.*

Did you say the equilibrium price rises by $\$\dfrac{8}{13}$ and the equilibrium quantity falls by $\dfrac{20}{13}$ units?

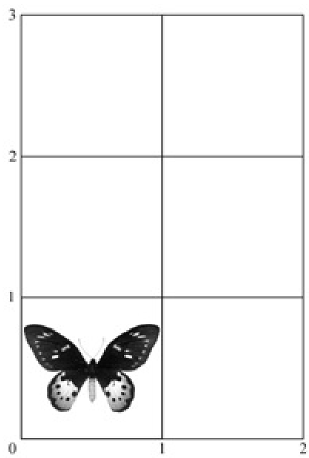
Notice that these figures are the second column of the inverse function $\begin{pmatrix} \dfrac{5}{13} & \dfrac{8}{13} \\ \dfrac{20}{13} & -\dfrac{20}{13} \end{pmatrix}$. The negative in the equilibrium output means it is falling.

Example

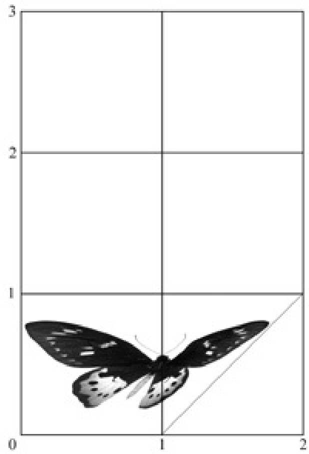
In the beginning of this module we said that matrices are used in geometry for translations, rotations and reflections. The following example shows a simple application of matrices in geometry which could be used in graphic art.

Have a look at a picture of a butterfly in the Cartesian coordinate system below.

We can change the shape of the picture by using a matrix to transform the shape.



If we want to distort this we could grab it at the point (1,1) and pull it across to (2,1), but keep the point (0,1) anchored. We could do this arithmetically using matrices. For example if we multiply the original point by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we will get the new point $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. This opens up a range of possibilities using computer graphics.



Activity 6.13

1. Suppose we were given a supply and demand model:

   Demand: $1p + 0.25q = 25$
   Supply: $1p - 0.2q = -2$

   The imposition of a tax changes the constant in the supply function from -2 to $t - 2$.

   (a) Change the supply function to include the tax changes

   (b) Using matrices, solve the two simultaneous equations with t as a variable in the solution. The inverse of the coefficient matrix is $\begin{pmatrix} \frac{4}{9} & \frac{5}{9} \\ \frac{20}{9} & \frac{-20}{9} \end{pmatrix}$

   (c) If there was no tax, what would the equilibrium price and quantity be?

   (d) If a tax of 1 unit was imposed what would the new equilibrium price and quantity be?

   (e) Complete the following sentence:

   For each unit of tax that is imposed the equilibrium price <u>*rises/falls*</u> *by___________ while the equilibrium quantity* <u>*rises/falls*</u> *by __________.*

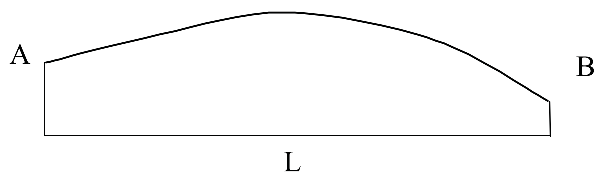
That's the end of this module. For most of you it may have been the first time you have been introduced to the concept of matrices. Matrices should help many of you in your future studies whether it be in the business, engineering or science field.

Before you are really finished, you should do a number of things:

1. Have a look at your action plan for study. Are you on schedule? Maybe you need to restructure your plan or contact your tutor to discuss any delays or concerns.
2. Make a summary of the important points in this module if you have not already done so. Have you added new words into your personal glossary?
3. Practice some more real world problems in 'a taste of things to come'
4. Check your skill level by attempting the post-test.
5. When you are ready, complete and submit your assignment.

6.6 A taste of things to come

1. Vertical curves for roads are often designed to have a parabolic cross-section as in the diagram below. The equation of these curves is the parabolic form $y = ax^2 + bx + c$



In laying out these curves surveyors use four pieces of information to determine the coefficients a, b and c.

- the elevation above the datum line at A where the curve begins,
- the gradient (G_1) of the road at A,
- the gradient (G_2) of the road at B where the curve ends,
- the horizontal length (L) of the curve from A to B.

Using this information surveyors develop the following formulas to find a, b and c in the general parabolic equation:

$$a = \frac{G_2 - G_1}{2L}$$

$$b = G_1$$

c = elevation at A

so that the equation of the parabola becomes

$$y = \left(\frac{G_2 - G_1}{2L}\right)x^2 + G_1 x + A$$

A road with a vertical curve of horizontal length 500 m passes through the three points given below where x is measured from A and y is height above the datum line.

$x = 100$ m, $y = 320$ m

$x = 200$ m, $y = 322$ m

$x = 400$ m, $y = 321$ m

(a) Using the formula: $y = \left(\frac{G_2 - G_1}{2L}\right)x^2 + G_1 x + A$, develop three equations in the three unknowns (G_1, G_2 and A)

(b) If the inverse of the matrix $\begin{pmatrix} 90 & 10 & 1 \\ 160 & 40 & 1 \\ 240 & 16 & 1 \end{pmatrix}$ is $\begin{pmatrix} -0.02 & 0.025 & -0.005 \\ 0.013 & -0.025 & 0.01167 \\ 2.667 & -2 & 0.333 \end{pmatrix}$,

Using matrices find:

(i) elevation at A,

(ii) the first gradient G_1,

(iii) the second gradient G_2,

(c) Using your knowledge of parabolas find the maximum elevation of the curve.

2. A logging company has a contract with a local mill to provide 1000 m^3 of hoop pine, 800 m^3 of radiata, and 600 m^3 of Cyprus per month. There are three regions available for logging. The following table gives the species mix, and timber density for each region.

Region	West	North	East
Hoop	70%	10%	5%
Radiata	20%	60%	20%
Cyprus	10%	30%	75%
Volume/hectare	330 m^3/ha	390 m^3/ha	290 m^3/ha

The logging company wants to know how many hectares should be logged in each operating region listed above to deliver exactly the required volume of logs. To find this do the following:

(a) From the table find the total volume/ha for each of the types of timber for each region.

(b) Convert this into a 3×3 matrix. Call this matrix A.

(c) Since we want to know the number of hectares to be logged in each region, let:

x = number of hectares logged in West region

y = number of hectares logged in North region

z = number of hectares logged in East region

Create a matrix equation $AX = C$ where X is a 3×1 matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and C is the 3×1 matrix showing the total amount of each type of timber to be logged per month.

(d) Solve the matrix equation for X and interpret the solution.

Hint: the inverse of the coefficient matrix (correct to 5 decimal places) is:

$$\begin{pmatrix} 0.00455 & -0.00070 & -0.00012 \\ -0.00128 & 0.00513 & -0.00128 \\ 0 & -0.00265 & 0.00531 \end{pmatrix}$$

6.7 Post-test

1. A person wishes to improve her fitness and decides to jog, cycle and walk. She finds that she can average speeds of 10 km per hour when jogging, 25 km per hour when cycling and 5 km per hour when walking. She keeps a diary of her training over five days as follows:

	Time (hours)				
	Day 1	Day 2	Day 3	Day 4	Day 5
Jogging	1	1.5	0	1	1
Cycling	0	0.5	1	1	1.5
Walking	1	1	2	1	0

(a) Express the entries of this diary as a matrix (A) with labels for the rows and columns.

(b) What is the order of this matrix?

(c) Express the speeds given above as a matrix (B) with one row.

(d) What does B × A represent?

(e) Perform the matrix multiplication B × A.

(f) In the second month of training, she increases her training schedule to prepare for a fun run to include morning as well as evening sessions. The morning sessions follow the same pattern as that given in matrix A. The evening session times are as follows:

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Construct a matrix T that gives the total times for each day.

(g) Use T to calculate the total distance covered by the athlete in month 2 (assuming 4 full weeks).

2. Find the values of the unknowns if the following matrices are equal.

$$\begin{pmatrix} x & y-2 & 2z \\ t+3 & 3r-1 & 5s \end{pmatrix} = \begin{pmatrix} 3 & 5 & 6 \\ 9 & 8 & 0 \end{pmatrix}$$

$x =$ $y =$ $z =$ $t =$ $r =$ $s =$

3. Given the following matrices:

$$A = \begin{pmatrix} 3 & -1 \\ -4 & 2 \\ 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 7 & 0 \\ 1 & 5 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 5 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Perform the following calculations (indicate where this is impossible):

(a) $2B + C$

(b) AC

(c) CB

(d) $(CB)^2$

4. If $2x + 3y = 10$ and $3x + 4y = 9$, then solve these simultaneous equations using matrices. (Hint the inverse of the coefficient matrix is $\begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$)

6.8 Solutions

Solutions to activities

Activity 6.1

1. (a) The rows represent the number of each type of shed and the columns represent the number sold in each of the first three months.

$$\begin{array}{r} \\ \text{small} \\ \text{medium} \\ \text{large} \end{array}
\begin{array}{c} \begin{array}{ccc} \text{J} & \text{F} & \text{M} \end{array} \\ \begin{pmatrix} 4 & 10 & 7 \\ 8 & 5 & 10 \\ 0 & 1 & 4 \end{pmatrix} \end{array}$$

(b) The rows represent the costs at different resorts and the columns represent the costs of different types of motel rooms.

$$\begin{array}{r} \\ \text{Sunny Resort} \\ \text{Sunny Surf} \\ \text{Shady Lane} \end{array}
\begin{array}{c} \begin{array}{cccc} \text{Lux.} & \text{F'ly} & \text{Sngle.} & \text{Dble.} \end{array} \\ \begin{pmatrix} 200 & 180 & 80 & 140 \\ 400 & 250 & 150 & 280 \\ 180 & 180 & 50 & 90 \end{pmatrix} \end{array}$$

2. Here are two examples:

	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
11082	1	1	0	1	0.5	2	3
11061	1	0	0	1	0	0	3
11034	2	2	0	2	0	0	1

$$\begin{array}{r} \\ \\ 82 \\ \text{Unit } 32 \\ 33 \end{array}
\begin{array}{c} \text{Days} \\ \begin{array}{ccccccc} \text{M} & \text{T} & \text{W} & \text{T} & \text{F} & \text{S} & \text{S} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0.5 & 2 & 3 \\ 1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 2 & 2 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
11082	1	1	1	1	1	1	0

$$\begin{array}{r} \\ \\ 82 \end{array}
\begin{array}{c} \text{Days} \\ \begin{array}{ccccccc} \text{M} & \text{T} & \text{W} & \text{T} & \text{F} & \text{S} & \text{S} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Activity 6.2

1.

	Definition	Example
(a) Matrix	An array of rows and columns	$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$
(b) Order of a matrix	The number of rows and columns in a matrix	The above is a 2×2 matrix
(c) Element of a matrix	The number inside a matrix specified by the row and column number	2 is an element of the above matrix
(d) $a_{i,j}$	The element in the i^{th} row and j^{th} column	$2_{1,1}$ is in the first row and first column
(e) Equal matrices	Equal when order is the same and corresponding elements are equal	$\begin{pmatrix} 2 & 5 \\ 3 & x \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & 9 \end{pmatrix}$ are equal when $x = 9$

2. $a_{2,1} = 0.4$. There is a 40% chance wheat prices will go up if the weather conditions are moderate.

3. (a)

$$\text{Ore Reserves (millions of tonnes)}$$
$$\begin{array}{ll} & \begin{array}{ccc} \text{Proved} & \text{Measured} & \text{Inferred} \end{array} \\ \text{Mine Sites} \begin{array}{r} \text{Matilda} \\ \text{Felicity} \\ \text{Mt Granny} \end{array} & \begin{pmatrix} 0.5 & 1 & 0.2 \\ 0 & 0.07 & 0.023 \\ 0 & 0 & 1.2 \end{pmatrix} \end{array}$$

(b)

$$\text{Gold Reserves (g/t)}$$
$$\begin{array}{ll} & \begin{array}{ccc} \text{Proved} & \text{Measured} & \text{Inferred} \end{array} \\ \text{Mine Sites} \begin{array}{r} \text{Matilda} \\ \text{Felicity} \\ \text{Mt Granny} \end{array} & \begin{pmatrix} 1.8 & 2.4 & 2.8 \\ 0 & 9.2 & 2.4 \\ 0 & 0 & 8.7 \end{pmatrix} \end{array}$$

4. (a) 1×3 matrix
(b) 5×1 matrix
(c) 5×2 matrix
(d) 2×2 matrix
(e) not a matrix (one element missing)

Activity 6.3

1. (a) $A + B = \begin{pmatrix} 0 & -4 \\ 2 & 8 \end{pmatrix}$

(b) not possible

(c) $W + H = \begin{pmatrix} \frac{11}{4} & -9 & 6 \\ 0 & -7 & 0 \\ 2 & 1 & \frac{14}{10} \end{pmatrix}$

(d) $L + C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. This is called the zero matrix.

(e) $S + R + T = \begin{pmatrix} 12 & 18 \\ 9 & 6 \end{pmatrix}$

(f) $V + G = \begin{pmatrix} 2 + x & 3y \\ 2x & 2y + 1 \end{pmatrix}$

2. (a) **Shopping centre matrix**

$$\begin{array}{ll} & \text{hair product} \\ & \begin{array}{ccc} \text{shp.} & \text{cndt.} & \text{gel} \end{array} \\ \text{brand} \begin{array}{r} \text{own brand} \\ \text{your style} \end{array} & \begin{pmatrix} 20 & 15 & 15 \\ 5 & 5 & 8 \end{pmatrix} \end{array}$$

Suburbs matrix

$$\begin{array}{ll} & \text{hair product} \\ & \begin{array}{ccc} \text{shp.} & \text{cndt.} & \text{gel} \end{array} \\ \text{brand} \begin{array}{r} \text{own brand} \\ \text{your style} \end{array} & \begin{pmatrix} 10 & 5 & 5 \\ 10 & 12 & 14 \end{pmatrix} \end{array}$$

(b) Matrices are both 2×3

(c) $\begin{pmatrix} 30 & 20 & 20 \\ 15 & 17 & 22 \end{pmatrix}$

This matrix tells us the total number of each hair product for both brands in both stores.

Activity 6.4

1. $A - B = \begin{pmatrix} 4 & -12 \\ 0 & 0 \end{pmatrix}$

2. (i) $A + B = \begin{pmatrix} 0 & -4 \\ 2 & 8 \end{pmatrix}$

   $B + A = \begin{pmatrix} 0 & -4 \\ 2 & 8 \end{pmatrix}$

   Therefore $A + B = B + A$

   (ii) $A - B = \begin{pmatrix} 4 & -12 \\ 0 & 0 \end{pmatrix}$

   $B - A = \begin{pmatrix} -4 & 12 \\ 0 & 0 \end{pmatrix}$

   Therefore $A - B \neq B - A$

Activity 6.5

1. (a) $\begin{pmatrix} 4 & -16 \\ 2 & 8 \end{pmatrix}$

   (b) $\begin{pmatrix} 6 & -12 \\ -3 & -12 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & -4 \\ -1 & -4 \end{pmatrix}$

2. $1.04 \times \begin{pmatrix} 1.8 & 2.4 & 2.8 \\ 0 & 9.2 & 2.4 \\ 0 & 0 & 8.7 \end{pmatrix} = \begin{pmatrix} 1.872 & 2.496 & 2.912 \\ 0 & 9.568 & 2.496 \\ 0 & 0 & 9.048 \end{pmatrix}$

3. $\frac{1}{3}\begin{pmatrix} 1 & 11 \\ 2 & -2 \end{pmatrix}$

Activity 6.6

(a) (16 + 4) = (20)

(b) (–9 + –3 + 20) = (8)

(c) not possible

Activity 6.7

1. (a) 1 × 2 and 2 × 3

   (b) 1 × 3

   (c) $\begin{pmatrix} 21 & -66 & 18 \end{pmatrix}$

2. The matrices are of order 1 × 3 and 3 × 2 so the resulting matrix will be of order 1 × 2
   $\begin{pmatrix} 27+1+1.2 & 72+0+.3 \end{pmatrix} = \begin{pmatrix} 29.2 & 72.3 \end{pmatrix}$

Activity 6.8

1. (a) 2 × 2 and 2 × 3

   (b) 2 × 3

   (c) $\begin{pmatrix} 3+18 & 15+-81 & 0+18 \\ 1+8 & 5+-36 & 0+8 \end{pmatrix} = \begin{pmatrix} 21 & -66 & 18 \\ 9 & -31 & 8 \end{pmatrix}$

2. The matrices are of order 2 × 3 and 3 × 2 so the resulting matrix will be of order 2 × 2

$$\begin{pmatrix} 27+1+1.2 & 72+0+0.3 \\ 12+12 & 32+0+3 \end{pmatrix} = \begin{pmatrix} 29.2 & 72.3 \\ 24 & 35 \end{pmatrix}$$

3. (a) $A \times B = \begin{pmatrix} 27 & 36 \\ 26 & 58 \end{pmatrix}$

   (b) $B \times A = \begin{pmatrix} 51 & 46 \\ 24 & 34 \end{pmatrix}$

   (c) $A \times B \neq B \times A$. The first row in the first matrix is multiplied by the first column (3 × 1 + 6 × 4). This is not the same when you swap the matrices (1 × 3 + 8 × 6).

4. (a) $A \times B = \begin{pmatrix} -4-8 & 8-32 \\ -2+4 & 4+16 \end{pmatrix} = \begin{pmatrix} -12 & -24 \\ 2 & 20 \end{pmatrix}$

   (b) $X \times Z$ not possible since the number of columns of X (3) is not the same as the number of rows of Z (1).

$$(c)\, W \times H = \begin{pmatrix} \frac{3}{2}+8-6 & -2-16-\frac{9}{4} & 18-0-\frac{3}{2} \\ \frac{3}{4}+9+0 & -1-18+0 & 9-0+0 \\ 0-\frac{1}{4}+\frac{9}{5} & 0+\frac{1}{2}+\frac{27}{40} & 0+0+\frac{9}{20} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{2} & \frac{-81}{4} & \frac{33}{2} \\ \frac{39}{4} & -19 & 9 \\ \frac{31}{20} & \frac{47}{40} & \frac{9}{20} \end{pmatrix}$$

(d) Not possible. The number of columns of L (2) does not equal the number of rows of C (3)

(e) $S \times R \times T = \begin{pmatrix} 12+0 & 6+30 \\ 12+0 & 6+5 \end{pmatrix} \times T = \begin{pmatrix} 12 & 36 \\ 12 & 11 \end{pmatrix} \times T$

$$= \begin{pmatrix} 12 & 36 \\ 12 & 11 \end{pmatrix} \begin{pmatrix} 4 & 9 \\ 7 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 48+252 & 108+0 \\ 48+77 & 108+0 \end{pmatrix}$$

$$= \begin{pmatrix} 300 & 108 \\ 125 & 108 \end{pmatrix}$$

(f) $V \times G = \begin{pmatrix} 2x+xy & 2xy+y \\ 2x+2yx & 2xy+2y \end{pmatrix}$

Activity 6.9

1. (a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if you are pre-multiplying or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ if you are post-multiplying.

2. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 9 \\ 9 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 9 & 1 \end{pmatrix}$

Activity 6.10

$$A \times B = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 6-5 & -2+2 \\ 15-15 & -5+6 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$B \times A = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So A and B are inverses of each other.

Activity 6.11

1. (a) $\begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$

   (b) First change it into standard form

   $$-2x + 1y = 4$$
   $$-2x + 5y = -3$$

   Now we can put it into matrix form

   $$\begin{pmatrix} -2 & 1 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

   (c) First put into standard form

   $$-3x + 1y - 2z = 0$$
   $$4x + 1y + 0z = 12$$
   $$3x + 9y + 4z = 9$$

   Now we can put into matrix form

   $$\begin{pmatrix} -3 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & 9 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \\ 9 \end{pmatrix}$$

Activity 6.12

1. The matrix equation is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

$$\begin{aligned}
\begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{10}{3} - \frac{4}{3} \\ \frac{-5}{3} + \frac{8}{3} \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
x = 2 &\text{ and } y = 1
\end{aligned}$$

Check your solution.

2. The matrix equation is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$$

To solve we must multiply both sides by the inverse:

$$\begin{aligned}
\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} -80 + 96 + 9 \\ 26 - 30 - 3 \\ 10 - 12 - 1 \end{pmatrix} \\
&= \begin{pmatrix} 25 \\ -7 \\ -3 \end{pmatrix}
\end{aligned}$$

So $x = 25$, $y = -7$ and $z = -3$

Check your solution.

3. $$P = \begin{pmatrix} \frac{1}{8} & \frac{-1}{8} \\ \frac{1}{2} & \frac{-1}{4} \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{8} + \frac{-1}{2} & \frac{3}{8} + \frac{-2}{8} \\ \frac{7}{2} + \frac{-4}{4} & \frac{3}{2} + \frac{-2}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{5}{2} & 1 \end{pmatrix}$$

Activity 6.13

(a) $1p + 0.25q = 25$

$1p - 0.2q = t - 2$

(b) $$\begin{pmatrix} 1 & 0.25 \\ 1 & -0.2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 25 \\ t-2 \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{4}{9} & \frac{5}{9} \\ \frac{20}{9} & \frac{-20}{9} \end{pmatrix} \begin{pmatrix} 25 \\ t-2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{100}{9} + \frac{5(t-2)}{9} \\ \frac{500}{9} + \frac{-20(t-2)}{9} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{100}{9} + \frac{5t-10}{9} \\ \frac{500}{9} + \frac{-20t+40}{9} \end{pmatrix}$$

(c) If there was no tax, the equilibrium price would be $\frac{100}{9} + \frac{-10}{9} = \frac{90}{9}$ or about \$10 and the new equilibrium quantity would be $\frac{500}{9} + \frac{40}{9} = \frac{540}{9}$ or about 60 units.

(d) A tax of one unit would give a equilibrium price of $\frac{100}{9} + \frac{-5}{9} = \frac{95}{9}$ or about \$10.6 and an equilibrium quantity of $\frac{500}{9} + \frac{20}{9} = \frac{520}{9}$ or about 57.8 units.

(e) For each unit of tax that is imposed the equilibrium price rises by \$$\frac{5}{9}$ while the equilibrium quantity falls by $\frac{20}{9}$ units.

Solutions to a taste of things to come

1. (a) Substitute the three sets of coordinates into $y=\dfrac{G_2-G_1}{2L}x^2+G_1x+A$ giving this set of equations.

$$10000\times\frac{G_2-G_1}{2\times 500}+100\times G_1+A=320$$

$$40000\times\frac{G_2-G_1}{2\times 500}+200\times G_1+A=322$$

$$160000\times\frac{G_2-G_1}{2\times 500}+400\times G_1+A=321$$

simplifying these equations gives:

$$10G_2-10G_1+100G_1+A=320$$
$$40G_2-40G_1+200G_1+A=322$$
$$160G_2-160G_1+400G_1+A=321$$

Collecting like terms gives:

$$90G_1+10G_2+A=320$$
$$160G_1+40G_2+A=322$$
$$240G_1+160G_2+A=321$$

(b) Putting into matrix form

$$\begin{pmatrix} 90 & 10 & 1 \\ 160 & 40 & 1 \\ 240 & 160 & 1 \end{pmatrix}\begin{pmatrix} G_1 \\ G_2 \\ A \end{pmatrix}=\begin{pmatrix} 320 \\ 322 \\ 321 \end{pmatrix}$$

$$\begin{pmatrix} -0.02\times 320+0.025\times 322-0.005\times 321 \\ 0.013\times 320+-0.025\times 322+0.01167\times 321 \\ 2.667\times 320+-2\times 322+0.33\times 321 \end{pmatrix}$$

$$\begin{pmatrix} 0.045 \\ -0.038 \\ 316.33 \end{pmatrix}$$

$$\begin{pmatrix} 0.045 \\ -0.1439 \\ 316.33 \end{pmatrix}$$

Solving gives $G_1 = 0.045$, $G_2 = -0.038$ and $A = 316.33$

(i) Elevation at $A = 316.33$ m.
(ii) $G_1 = 0.045$
(iii) $G_2 = -0.038$

(d) The parabola has a maximum at $x = \frac{-b}{2a}$ so we need to find a and b.

$$b = 0.045$$

$$\begin{aligned} a &= \frac{G_2 - G_1}{2L} \\ &= \frac{-0.038 - 0.045}{1000} \\ &= -0.000083 \end{aligned}$$

So the parabola has maximum when $x = \frac{-0.045}{-0.000166}$

$$x = 271.084$$

Substituting this value of x into the equation for the parabola gives

$$y = -0.000083(271.084)^2 + 0.045(271.084) + 316.33$$

$$= 322.429 \text{ m.}$$

So the curve has a maximum elevation of about 322 metres.

2. (a)

Region	West	North	East
Hoop	231	39	14.5
Radiata	66	234	58
Cyprus	33	117	217.5

(b) $A = \begin{pmatrix} 231 & 39 & 14.5 \\ 66 & 234 & 58 \\ 33 & 117 & 217.5 \end{pmatrix}$

(c) Since we want to know the number of hectares that should be logged in each region, let

x = number of hectares logged in West region
y = number of hectares logged in North region
z = number of hectares logged in East region

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $C = \begin{pmatrix} 1000 \\ 800 \\ 600 \end{pmatrix}$

Then $A \times C$ becomes:

$$\begin{pmatrix} 231 & 39 & 14.5 \\ 66 & 234 & 58 \\ 33 & 117 & 217.5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1000 \\ 800 \\ 600 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 231 & 39 & 14.5 \\ 66 & 234 & 58 \\ 33 & 117 & 217.5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1000 \\ 800 \\ 600 \end{pmatrix}$$

$$\begin{pmatrix} 0.00455 & -0.00070 & -0.00012 \\ -0.00128 & 0.00468 & -0.00128 \\ 0 & -0.00265 & 0.00531 \end{pmatrix}\begin{pmatrix} 231 & 39 & 14.5 \\ 66 & 234 & 58 \\ 33 & 117 & 217.5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.00455 & -0.00070 & -0.00012 \\ -0.00128 & 0.00468 & -0.00128 \\ 0 & -0.00265 & 0.00531 \end{pmatrix}\begin{pmatrix} 1000 \\ 800 \\ 600 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4.55 - 0.56 - 0.072 \\ -1.28 + 4.104 - 0.768 \\ 0 - 2.12 + 3.186 \end{pmatrix}$$

$$= \begin{pmatrix} 3.92 \\ 2.06 \\ 1.07 \end{pmatrix}$$

$x = 3.92$

$y = 2.06$

$z = 1.07$

Answer:

To fulfil the contract the company needs to log 3.92 hectares in the West region, 2.06 hectares in the North region and 1.07 hectares in the East region.

Solutions to post-test

1. (a)

$$A = \begin{array}{r} \\ \text{Jogging} \\ \text{Cycling} \\ \text{Walking} \end{array} \begin{array}{c} \text{Day} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 1.5 & 0 & 1 & 1 \\ 0 & 0.5 & 1 & 1 & 1.5 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix} \end{array}$$

(b) order of matrix = 3×5

(c) $B = (10 \quad 25 \quad 5)$

(d) it represents the number of kilometres travelled every day for the week.

(e)

$$(10 \quad 25 \quad 5)\begin{pmatrix} 1 & 1.5 & 0 & 1 & 1 \\ 0 & 0.5 & 1 & 1 & 1.5 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$= (10+0+5 \quad 15+12.5+5 \quad 0+25+10 \quad 10+25+5 \quad 10+37.5+0)$$

$$= (15 \quad 32.5 \quad 35 \quad 40 \quad 47.5)$$

(f) $T = A + E = \begin{pmatrix} 2 & 2.5 & 1 & 2 & 2 \\ 1 & 1.5 & 1 & 3 & 1.5 \\ 2 & 1 & 3 & 2 & 1 \end{pmatrix}$

(g) Times for the month

$$= 4 \times \left(\begin{pmatrix} 1 & 1.5 & 0 & 1 & 1 \\ 0 & 0.5 & 1 & 1 & 1.5 \\ 2 & 1 & 2 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix} \right)$$

$$= 4 \times \begin{pmatrix} 2 & 2.5 & 1 & 2 & 2 \\ 1 & 1.5 & 1 & 3 & 1.5 \\ 3 & 1 & 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 10 & 4 & 8 & 8 \\ 4 & 6 & 4 & 12 & 6 \\ 12 & 4 & 12 & 8 & 4 \end{pmatrix}$$

$$\text{Distance} = (10 \quad 25 \quad 5)\begin{pmatrix} 8 & 10 & 4 & 8 & 8 \\ 4 & 6 & 4 & 12 & 6 \\ 8 & 4 & 12 & 8 & 4 \end{pmatrix}$$

$$= (80+100+40 \quad 100+150+20 \quad 40+100+60 \quad 80+300+40 \quad 80+150+20)$$

$$= (220 \quad 270 \quad 200 \quad 420 \quad 250)$$

2. $x = 3$; $y = 7$; $z = 3$; $t = 6$; $r = 3$; $s = 0$

3. (a) $2B+C$ is not possible since the matrices are not of the same order.

   (b) $A \times C$ is not possible since the number of columns of *A* is not the same as the number of rows of *C*.

   (c)

$$CB = \begin{pmatrix} 2+5 & 14+25 & 0-5 \\ 2+1 & 14+5 & 0-1 \\ 1+2 & 7+10 & 0-2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 39 & -5 \\ 3 & 19 & -1 \\ 3 & 17 & -2 \end{pmatrix}$$

   (d)

$$CB^2 = \begin{pmatrix} 7 & 39 & -5 \\ 3 & 19 & -1 \\ 3 & 17 & -2 \end{pmatrix}\begin{pmatrix} 7 & 39 & -5 \\ 3 & 19 & -1 \\ 3 & 17 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 49+117-15 & 273+741-85 & -35-39+10 \\ 21+57-3 & 117+361-17 & -15-19+2 \\ 21+51-6 & 117+323-34 & -15-17+4 \end{pmatrix}$$

$$= \begin{pmatrix} 151 & 929 & -64 \\ 75 & 461 & -32 \\ 66 & 406 & -28 \end{pmatrix}$$

4. $2x + 3y = 10$
   $3x + 4y = 9$

   In matrix form,

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 10 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -13 \\ 12 \end{pmatrix}$$

   $x = -13$, $y = 12$

   Check by substituting back into original equations.